

EXISTENCE OF FIXED POINTS FOR CONDENSING OPERATORS UNDER AN INTEGRAL CONDITION

VATAN KARAKAYA, NOUR EL HOUDA BOUZARA, KADRI DOGAN,
AND YUNUS ATALAN

ABSTRACT. Our aim in this paper is to present results of existence of fixed points for continuous operators in Banach spaces using measure of noncompactness under an integral condition. This results are generalization of results given by A. Aghajania and M. Aliaskaria in [2] which are generalization of Darbo's fixed point theorem. As application we use these results to solve an integral equations in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The Darbo's fixed point theorem which guarantees the existence of fixed point for so called condensing mappings is very famous, since it generalizes two important theorems : The Banach principle theorem and The classical Schauder theorem. Over recent years, many authors presented works that give generalization of this theorem, see Aghajani et al. in [4], [2], [3], Samadi and Ghaemi in [11], [12] and others.

Throughout this paper, X is assumed to be a Banach space and $BC(\mathbb{R}^+)$ is the space of all real functions defined, bounded and continuous on \mathbb{R}^+ . The family of bounded subset, closure and closed convex hull of X are denoted by \mathcal{B}_X , \overline{X} and $ConvX$, respectively.

Definition 1. [6] Let X be a Banach space and \mathcal{B}_X the family of bounded subset of X . A map

$$\mu : \mathcal{B}_X \rightarrow [0, \infty)$$

is called measure of noncompactness defined on X if it satisfies the following:

- (1) $\mu(A) = 0 \Leftrightarrow A$ is a precompact set.
- (2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
- (3) $\mu(A) = \mu(\overline{A})$, $\forall A \in \mathcal{B}_X$.
- (4) $\mu(ConvA) = \mu(A)$.
- (5) $\mu(\lambda A + (1 - \lambda)B) \leq \lambda \mu(A) + (1 - \lambda) \mu(B)$, for $\lambda \in [0, 1]$.
- (6) Let (A_n) be a sequence of closed sets from \mathcal{B}_X such that $A_{n+1} \subseteq A_n$, $(n \geq 1)$ and $\lim_{n \rightarrow \infty} \mu(A_n) = 0$, then the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and A_∞ is precompact.

Definition 2. A summable function is function for which the integral exists and is finite.

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Definition 3. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a point $x_0 \in \mathbb{R}$. The function f is said to be upper (resp. lower) semi-continuous at the point x_0 if

$$f(x_0) \geq \limsup_{x \rightarrow x_0} f(x) \quad (\text{resp. } f(x_0) \leq \liminf_{x \rightarrow x_0} f(x)).$$

Lemma 1. [4] Let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nondecreasing and upper semi-continuous function. Then,

$$\lim_{n \rightarrow \infty} \psi^n(t) = 0 \text{ for each } t > 0 \Leftrightarrow \psi(t) < t \text{ for any } t > 0.$$

Theorem 1 (Banach contraction theorem). [1] Let X be a Banach space and $T : X \rightarrow X$ be a contraction mapping on X , i.e. there is a nonnegative real number $k < 1$ such that

$$\|Tx - Ty\| \leq k \|x - y\| \quad \forall x, y \in X.$$

Then the map T admits one and only one fixed point x^* in X .

Theorem 2 (Schauder theorem). [1] Let A be a nonempty, convex, compact subset of a Banach space X and suppose $T : A \rightarrow A$ is continuous. Then T has a fixed point.

The famous Darbo's theorem is as following

Theorem 3 (Darbo's theorem). [7] Let C be a nonempty closed, bounded and convex subset of X . If $T : C \rightarrow C$ is a continuous mapping

$$\mu(TA) \leq k\mu(A), \quad k \in [0, 1],$$

then T has a fixed point.

Theorem 4. [2] Let X be a Banach space and A be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : A \rightarrow A$ be a continuous operator which satisfies the following inequality

$$\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \leq \Psi \left(\int_0^{\mu(X)} \varphi(\gamma) d\gamma \right),$$

where μ is a measure of noncompactness, $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a nondecreasing function such that $\lim_{n \rightarrow \infty} \Psi^n(t) = 0$, $\forall t \geq 0$ and $\varphi : [0, +\infty[\rightarrow [0, +\infty[$ is integral mapping which is summable on each compact subset of $[0, +\infty[$ and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(\gamma) d\gamma > 0$.

Then T has at least one fixed point in X .

2. MAIN RESULTS

Theorem 5. Let X be a Banach space and A be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : A \rightarrow A$ be a continuous operator which satisfies the following inequality

$$(2.1) \quad \Phi \left(\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\mu(X)} \varphi(\gamma) d\gamma \right),$$

where μ is a measure of noncompactness and

- (i) $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing and concave function such that $\lim_{n \rightarrow \infty} \Psi^n(t) = 0$, $\forall t \geq 0$.

- (ii) $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing subadditive function such that $\Phi(t) \geq t$ and $\lim_{n \rightarrow \infty} \Phi(x_n) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} x_n = 0$.
- (iii) $\varphi : [0, +\infty[\rightarrow [0, +\infty[$ is an integral mapping which is summable on each compact subset of $[0, +\infty[$ and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(\omega) d\omega > 0$.

Then T has at least one fixed point in X .

Proof. Consider $(A_n)_{n=0}^\infty$, a closed and convex sequence of subset of X such that $A_{n+1} = \text{Conv}(TA_n)$. We notice that $A_1 = \text{Conv}(TA_0) \subseteq A_0$ and $A_2 = \text{Conv}(TA_1) \subseteq A_1$. By induction, we get

$$\dots A_{n+1} \subseteq A_n \subseteq \dots \subseteq A_0.$$

In further, we have

$$\begin{aligned} \int_0^{\mu(A_{n+1})} \varphi(\gamma) d\gamma &= \int_0^{\mu(\text{Conv}(TA_n))} \varphi(\gamma) d\gamma \\ &= \int_0^{\mu(TA_n)} \varphi(\gamma) d\gamma. \end{aligned}$$

Using (2.1), we get

$$(2.2) \quad \Phi\left(\int_0^{\mu(A_{n+1})} \varphi(\gamma) d\gamma\right) = \Phi\left(\int_0^{\mu(TA_n)} \varphi(\gamma) d\gamma\right) \leq \Psi\left(\int_0^{\mu(A_n)} \varphi(\gamma) d\gamma\right).$$

Moreover, using $\Phi(t) \geq t$ we get

$$\int_0^{\mu(A_n)} \varphi(\gamma) d\gamma \leq \Phi\left(\int_0^{\mu(A_n)} \varphi(\gamma) d\gamma\right),$$

and

$$\begin{aligned} \int_0^{\mu(A_n)} \varphi(\gamma) d\gamma &= \int_0^{\mu(\text{Conv}(TA_{n-1}))} \varphi(\gamma) d\gamma \\ &= \int_0^{\mu(TA_{n-1})} \varphi(\gamma) d\gamma. \end{aligned}$$

Then,

$$\begin{aligned} \Psi\left(\int_0^{\mu(A_n)} \varphi(\gamma) d\gamma\right) &= \Psi\left(\int_0^{\mu(TA_{n-1})} \varphi(\gamma) d\gamma\right) \\ &\leq \Psi\left(\Phi\left(\int_0^{\mu(TA_{n-1})} \varphi(\gamma) d\gamma\right)\right) \\ &\leq \Psi^2\left(\int_0^{\mu(A_{n-1})} \varphi(\gamma) d\gamma\right). \end{aligned}$$

Repeating this process n times we get

$$\Phi\left(\int_0^{\mu(A_{n+1})} \varphi(\gamma) d\gamma\right) \leq \Psi^{n+1}\left(\int_0^{\mu(A_0)} \varphi(\gamma) d\gamma\right).$$

In view of condition (i), we get $\lim_{n \rightarrow \infty} \Psi^{n+1} \left(\int_0^{\mu(A_0)} \varphi(\gamma) d\gamma \right) = 0$. Then,

$$\lim_{n \rightarrow \infty} \Phi \left(\int_0^{\mu(A_{n+1})} \varphi(\gamma) d\gamma \right) = 0,$$

using (ii) we obtain

$$\lim_{n \rightarrow \infty} \int_0^{\mu(A_{n+1})} \varphi(\gamma) d\gamma = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \mu(A_{n+1}) = 0.$$

Consequently, A_∞ is compact and then T has at least one fixed point. \square

Remark 1. (i) $\Phi(x) = x$, then $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing additive (hence subadditive) function and $\lim_{n \rightarrow \infty} \Phi(x_n) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} x_n = 0$. Thus, $\Phi(x) = x$ satisfies condition (ii) of theorem (5) and inequality (2.1) will be

$$\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \leq \Psi \left(\int_0^{\mu(X)} \varphi(\gamma) d\gamma \right),$$

which is the condition given by Aghajani and Aliaskari in [2].

(ii) $\Phi(x) = x$, and $\Psi(x) = kx$, where $k \in [0, 1[$. Since $\lim_{n \rightarrow \infty} \Psi^n(x) = \lim_{n \rightarrow \infty} k^n x = 0$. Then, Ψ and Φ satisfy conditions (i) and (ii) of theorem (5). Then, inequality (2.1) become

$$\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \leq k \int_0^{\mu(X)} \varphi(\gamma) d\gamma,$$

which is a generalization of the result given by Branciari in [9].

(iii) $\Phi(x) = x$, $\Psi(x) = kx$ and $\varphi(x) = 1$. These functions satisfy conditions (i) – (iii) of theorem (5) and instead of inequality (2.1) we obtain the following inequality

$$\mu(TX) \leq k\mu(X),$$

which is the condition given by Darbo in his famous fixed point theorem (see [7]).

3. APPLICATIONS

In this section we use Theorem 5 to study the resolvability of the following integral equation in the Banach space $BC(\mathbb{R}^+)$ under more general hypothesis.

$$(3.1) \quad x(t) = f \left(t, \int_0^t g(t, s, x(s)) ds, x(t) \right), \quad t \in \mathbb{R}^+.$$

In what follow we formulate the assumptions under which equation (3.1) will be studied:

- (i) The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and $|f(t, x, 0)| \in BC(\mathbb{R}^+)$ for $t \in \mathbb{R}^+$, and $x \in \mathbb{R}$.

- (ii) There exist a nondecreasing, concave and upper semi-continuous function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and a nondecreasing, upper semi-continuous and sub-additive function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\Phi(t) \geq t$ and $\lim_{n \rightarrow \infty} \phi(x_n) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} x_n = 0$, for which the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the conditions

$$\begin{aligned} \Phi \left(\int^{|f(t,x,y_1)-f(t,x,y_2)|} \varphi(\gamma) d\gamma \right) &\leq \Psi \left(\int^{|y_1-y_2|} \varphi(\gamma) d\gamma \right), \\ \Phi \left(\int^{|f(t,x_1,y)-f(t,x_2,y)|} \varphi(\gamma) d\gamma \right) &\leq \Psi \left(\int^{|x_1-x_2|} \varphi(\gamma) d\gamma \right), \end{aligned}$$

where $\varphi : [0, \infty[\rightarrow [0, \infty[$ is summable on every compact subset of $[0, \infty[$ and for every $\epsilon > 0$, $\int_0^\epsilon \varphi(\omega) d\omega > 0$.

- (iii) The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{t \rightarrow \infty} a(t) = 0$, $b \in L_1(\mathbb{R}^+)$ and $|g(t, s, x)| \leq a(t)b(s)$ for $t, s \in \mathbb{R}^+$ such that $s \leq t$ and each $x \in \mathbb{R}$.
- (iv) There exists at least one positive constant r_0 such that the following inequality holds,

$$\int_0^r \varphi(\gamma) d\gamma \leq \int_0^r \varphi(\gamma) d\gamma + M_0 + M_1,$$

where $M_0 = \sup \left\{ \int_0^{|a(t)|} \int_0^t |b(s)| ds \varphi(\gamma) d\gamma \right\}$ and $M_1 = \Phi \left(\int_0^{\sup |f(t,0,0)|} \varphi(\gamma) d\gamma \right)$.

Theorem 6. *Under the hypothesis (i) – (iv) the integral equation (3.1) has at least one solution in the space $BC(\mathbb{R}^+)$.*

Proof. Study the solvability of equation (3.1) is equivalent to study the existence of fixed points of the following operator

$$Tx(t) = f \left(t, \int_0^t g(t, s, x(s)) ds, x(t) \right), \quad t \in \mathbb{R}^+.$$

For that we need to verify that under assumptions (i) – (iv) the operator T satisfies the conditions of Theorem 5.

First, let recall the following notions, the measure of noncompactness for a positive fixed t on $\mathcal{B}_{BC}(\mathbb{R}^+)$ is given by the following

$$\mu(X) = \omega_0(X) + \limsup_{t \rightarrow \infty} \text{diam} X(t),$$

where,

$$\text{diam} X(t) = \sup \{|x(t) - y(t)| : x, y \in X\}, \quad X(t) = \{x(t) : x \in X\},$$

and

$$\omega_0(X) = \lim_{L \rightarrow \infty} \omega_0^L(X).$$

$$\omega_0^L(X) = \lim_{\epsilon \rightarrow 0} \omega^L(X, \epsilon),$$

$$\omega^L(X, \epsilon) = \sup \{\omega^L(x, \epsilon) : x \in X\},$$

$$\omega^L(x, \epsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, L], |t - s| \leq \epsilon\}, \text{ for } L > 0.$$

To show that T is self-mappings, that is, T map a ball B_{r_0} into itself, let

$$\begin{aligned}
\int_0^{|Tx(t)|} \varphi(\gamma) d\gamma &\leq \Phi \left(\int_0^{|Tx(t)|} \varphi(\gamma) d\gamma \right) \\
&= \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t))|} \varphi(\gamma) d\gamma \right) \\
&\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(t, \int_0^t g(t,s,x(s))ds, 0)| + |f(t, \int_0^t g(t,s,x(s))ds, 0) - f(t, 0, 0)| + |f(t, 0, 0)|} \varphi(\gamma) d\gamma \right) \\
&\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(t, \int_0^t g(t,s,x(s))ds, 0)|} \varphi(\gamma) d\gamma \right) \\
&\quad + \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, 0) - f(t, 0, 0)|} \varphi(\gamma) d\gamma \right) + \Phi \left(\int_0^{|f(t, 0, 0)|} \varphi(\gamma) d\gamma \right).
\end{aligned}$$

Using (ii), we get

$$\int_0^{|Tx(t)|} \varphi(\omega) d\omega \leq \Psi \left(\int_0^{|x(t)|} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\left| \int_0^t g(t,s,x(s))ds \right|} \varphi(\gamma) d\gamma \right) + \Phi \left(\int_0^{\sup_t |f(t, 0, 0)|} \varphi(\gamma) d\gamma \right)$$

In view of condition (iii) and Lemma 1, we obtain

$$\begin{aligned}
\int_0^{|Tx(t)|} \varphi(\gamma) d\gamma &\leq \int_0^{|x(t)|} \varphi(\gamma) d\gamma + \int_0^{|a(t)| \int_0^t |b(s)| ds} \varphi(\gamma) d\gamma + M_1 \\
&\leq \int_0^{|x(t)|} \varphi(\gamma) d\gamma + M_0 + M_1.
\end{aligned}$$

Finally, the assumption (iv) guaranties the existence of a constant r_0 such that $TB_{r_0} \subseteq B_{r_0}$.

Now, let show that T satisfies Condition 2.1 of Theorem 5.

$$\begin{aligned}
\Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) &\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(t, \int_0^t g(t,s,y(s))ds, y(t))|} \varphi(\gamma) d\gamma \right) \\
&\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(t, \int_0^t g(t,s,y(s))ds, x(t))|} \varphi(\gamma) d\gamma \right) \\
&\quad + \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,y(s))ds, x(t)) - f(t, \int_0^t g(t,s,y(s))ds, y(t))|} \varphi(\gamma) d\gamma \right).
\end{aligned}$$

Using assumption (ii), we get

$$(3.2) \quad \Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x(t) - y(t)|} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\left| \int_0^t |g(t,s,x(s)) - g(t,s,y(s))| ds \right|} \varphi(\gamma) d\gamma \right)$$

Moreover,

$$\begin{aligned}
\Psi \left(\int_0^t |g(t, s, x(s)) - g(t, s, y(s))| ds \varphi(\gamma) d\gamma \right) &\leq \Psi \left(\int_0^t |g(t, s, x(s)) - g(t, s, y(s))| ds \varphi(\gamma) d\gamma \right) \\
&< \int_0^t |g(t, s, x(s)) - g(t, s, y(s))| ds \varphi(\gamma) d\gamma \\
&\leq \int_0^t |g(t, s, x(s))| ds + \int_0^t |g(t, s, y(s))| ds \varphi(\gamma) d\gamma \\
&\leq \int_0^t |g(t, s, x(s))| ds + \int_0^t |g(t, s, y(s))| ds \varphi(\gamma) d\gamma \\
&\leq \int_0^{2|a(t)|} |b(s)| ds \varphi(\gamma) d\gamma.
\end{aligned}$$

Since $\lim_{t \rightarrow \infty} a(t) = 0$ and $b \in L_1(\mathbb{R}^+)$, then for $t \geq L$ (where L is positive constant) we have

$$\int_0^{2|a(t)|} |b(s)| ds \varphi(\gamma) d\gamma \leq \epsilon,$$

where ϵ is an arbitrary positive number.

Consequently, Inequality 3.2 will be

$$\Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x(t) - y(t)|} \varphi(\gamma) d\gamma \right).$$

For $t \leq L$, we have

$$\omega_1^L(g, \epsilon) = \sup \left\{ \int_0^t |g(t, s, x(s)) - g(t, s, y(s))| ds : t, s \in [0, L], x, y \in B_{r_0} \text{ and } \|x - y\| \leq \epsilon \right\}.$$

Thus for $t \in [0, L]$, we obtain

$$\Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x(t) - y(t)|} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\omega_1^L(g, \epsilon)} \varphi(\gamma) d\gamma \right).$$

Since g is continuous, it is uniformly continuous on $[0, L] \times [0, L] \times [-r_0, r_0]$. Then,

$$\lim_{\epsilon \rightarrow 0} \omega_1^L(g, \epsilon) = 0.$$

Consequently, Inequality 3.2 will be

$$\Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x(t) - y(t)|} \varphi(\gamma) d\gamma \right).$$

Thus, for every $t \geq 0$ we have

$$\Phi \left(\int_0^{|Tx(t) - Ty(t)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x(t) - y(t)|} \varphi(\gamma) d\gamma \right),$$

hence,

$$\Phi \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(TX(t))} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(X(t))} \varphi(\gamma) d\gamma \right).$$

Now, let consider

$$\begin{aligned}
\Phi \left(\int_0^{|Tx(t)-Tx(l)|} \varphi(\gamma) d\gamma \right) &= \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(l, \int_0^l g(l,s,x(s))ds, x(l))|} \varphi(\gamma) d\gamma \right) \\
&\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(l, \int_0^t g(t,s,x(s))ds, x(t))|} \varphi(\gamma) d\gamma \right) \\
&\quad + \Phi \left(\int_0^{|f(l, \int_0^t g(t,s,x(s))ds, x(t)) - f(l, \int_0^l g(l,s,x(s))ds, x(t))|} \varphi(\gamma) d\gamma \right) \\
&\quad + \Phi \left(\int_0^{|f(l, \int_0^l g(l,s,x(s))ds, x(t)) - f(l, \int_0^l g(l,s,x(s))ds, x(l))|} \varphi(\gamma) d\gamma \right) \\
&\leq \Phi \left(\int_0^{|f(t, \int_0^t g(t,s,x(s))ds, x(t)) - f(l, \int_0^t g(t,s,x(s))ds, x(t))|} \varphi(\gamma) d\gamma \right) \\
&\quad + \Psi \left(\int_0^{\int_0^t |g(t,s,x(s)) - g(l,s,x(s))| ds} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{|x(t)-x(l)|} \varphi(\gamma) d\gamma \right).
\end{aligned}$$

Putting,

$$\begin{aligned}
\omega^L(gx, \epsilon) &= \sup \{ |g(t, s, x(s)) - g(l, s, x(s))| : l, t, s \in [0, L], x \in B_{r_0} \text{ and } |t-l| \leq \epsilon \}, \\
\omega^L(fx, \epsilon) &= \sup \{ |f(t, x, y) - f(l, x, y)| : l, t \in [0, L], x, y \in B_{r_0} \text{ and } |t-l| \leq \epsilon \}, \\
\omega^L(Tx, \epsilon) &= \sup \{ |Tx(t) - Tx(l)| : l, t \in [0, L], x \in B_{r_0} \text{ and } |t-l| \leq \epsilon \},
\end{aligned}$$

we get

$$\Phi \left(\int_0^{\omega^L(Tx, \epsilon)} \varphi(\gamma) d\gamma \right) \leq \Phi \left(\int_0^{\omega^L(fx, \epsilon)} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\omega^L(gx, \epsilon)} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\omega^L(x, \epsilon)} \varphi(\gamma) d\gamma \right).$$

We know that g is uniformly continuous on $[0, L] \times [0, L] \times [-r_0, r_0]$ and f is uniformly continuous on $[0, L] \times [-r_0, r_0] \times [-r_0, r_0]$, then we get

$$\Phi \left(\int_0^{\omega^L(Tx, \epsilon)} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\omega^L(x, \epsilon)} \varphi(\gamma) d\gamma \right).$$

and then,

$$\Phi \left(\int_0^{\omega^L(TX, \epsilon)} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\omega^L(X, \epsilon)} \varphi(\gamma) d\gamma \right).$$

By taking $\epsilon \rightarrow 0$, $L \rightarrow \infty$ and using the fact that Φ and Ψ are semicontinuous, we obtain

$$\Phi \left(\int_0^{\omega_0^L(TX)} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\omega_0^L(X)} \varphi(\gamma) d\gamma \right).$$

Finally,

$$\begin{aligned}
\Phi \left(\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \right) &\leq \Phi \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(TX(t))} \varphi(\gamma) d\gamma \right) + \Phi \left(\int_0^{\omega_0^L(TX)} \varphi(\gamma) d\gamma \right) \\
&\leq \Psi \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(X(t))} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\omega_0^L(X)} \varphi(\gamma) d\gamma \right)
\end{aligned}$$

using the fact that Ψ is concave, we get

$$\begin{aligned}
& \Psi \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(X(t))} \varphi(\gamma) d\gamma \right) + \Psi \left(\int_0^{\omega_0^L(X)} \varphi(\gamma) d\gamma \right) \\
= & \Psi \left(\frac{1}{2} \int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(X(t))} 2\varphi(\gamma) d\gamma \right) + \Psi \left(\frac{1}{2} \int_0^{\omega_0^L(X)} 2\varphi(\gamma) d\gamma \right) \\
\leq & \Psi \left(\frac{1}{2} \left(\int_0^{\limsup_{t \rightarrow \infty} \text{Diam}(X(t))} 2\varphi(\gamma) d\gamma + \int_0^{\omega_0^L(X)} 2\varphi(\gamma) d\gamma \right) \right) \\
= & \Psi \left(\frac{1}{2} \int_0^{\mu(TX)} 2\varphi(\gamma) d\gamma \right)
\end{aligned}$$

Finally, we obtain

$$\Phi \left(\int_0^{\mu(TX)} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{\mu(X)} \varphi(\gamma) d\gamma \right),$$

and T has a fixed point. \square

Example 1. *let the following integral equation*

$$(3.3) \quad x(t) = \sin t + \ln \left(1 + \int_0^t \frac{1}{t^2 + 1} e^{-s^2} \cos x(t) ds \right) + \ln(1 + x(t)).$$

Considering that by putting

$$f(t, x, y) = \sin t + \ln(1 + x) + \ln(1 + y)$$

and

$$g(t, s, x) = \frac{1}{t^2 + 1} e^{-s^2} \cos x,$$

we obtain an equation of the form (3.1) and it satisfies assumptions (i – v). Indeed, it is to see that assumption (i) is satisfied and by taking $\Phi(x) = kx$ for $k \in [0, 1[$, $\Psi(x) = \ln(1 + x)$ and $\varphi(t) = 1$, we get

$$\begin{aligned}
\Phi \left(\int_0^{|f(t, x, y_1) - f(t, x, y_2)|} \varphi(\gamma) d\gamma \right) &= k |f(t, x, y_1) - k f(t, x, y_2)| \\
&= k \ln \left(\frac{1 - |y_1|}{1 - |y_2|} \right) \\
&= k \ln \left(1 + \frac{1 + (|y_2| - |y_1|)}{1 - |y_2|} \right) \\
&\leq k \ln(1 + (|y_2| - |y_1|)) \\
&\leq \ln(1 + |y_2 - y_1|) \\
&= \Psi(|y_2 - y_1|).
\end{aligned}$$

Hence,

$$\Phi \left(\int_0^{|f(t, x, y_1) - f(t, x, y_2)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|y_1 - y_2|} \varphi(\gamma) d\gamma \right).$$

The same way we prove that,

$$\Phi \left(\int_0^{|f(t,x_1,y)-f(t,x_2,y)|} \varphi(\gamma) d\gamma \right) \leq \Psi \left(\int_0^{|x_1-x_2|} \varphi(\gamma) d\gamma \right).$$

Then, assumption (ii) holds.

In further,

$$\begin{aligned} |g(t, s, x)| &= \left| \frac{1}{t^2+1} e^{-s^2} \sin x \right| \\ &\leq \left| \frac{1}{t^2+1} e^{-s^2} \right|, \end{aligned}$$

then by taking $a(t) = \frac{1}{t^2+1}$ and $b(s) = e^{-s^2}$, it is easy to see that g , a and b satisfies the conditions in (iii).

Finally,

$$\int_0^r \varphi(\gamma) d\gamma \leq \int_0^r \varphi(\gamma) d\gamma + M_0 + M_1,$$

where $M_0 = \sup \left\{ \int_0^{|a(t)|} \int_0^{|b(s)|} \varphi(\gamma) d\gamma ds \right\}$ and $M_1 = \Phi \left(\int_0^{\sup |f(t,0,0)|} \varphi(\gamma) d\gamma \right)$, holds for every positive r .

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DEPARTMENT OF MATHEMATICAL ENGINEERING, FACULTY OF CHEMISTRY-METALLURGICAL, YILDIZ
TECHNICAL UNIVERSITY, ISTANBUL, TURKEY

E-mail address: `vkaya@yahoo.com`

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LETTERS, YILDIZ TECHNICAL UNI-
VERSITY, ISTANBUL, TURKEY

E-mail address: `bzr.nour@gmail.com`

DEPARTMENT OF MATHEMATICAL ENGINEERING, FACULTY OF CHEMISTRY-METALLURGICAL, YILDIZ
TECHNICAL UNIVERSITY, ISTANBUL, TURKEY

E-mail address: `dogankadri@hotmail.com`

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LETTERS, YILDIZ TECHNICAL UNI-
VERSITY, ISTANBUL, TURKEY

E-mail address: `yunus_atalan@hotmail.com`